Journal of Pure and Applied Algebra 26 (1982) 325–330
North-Holland Publishing Company

REDUCED POWERS, ULTRAPOWERS AND EXACTNESS OF LIMITS

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Communicated by C.J. Mulvey
Received 24 November 1980

It is well known that on inverse systems of linearly compact modules the inverse limit functor is exact (Jensen [6, §7]); linearly compact modules are equationally compact, [7], but on systems of equationally compact modules $\varprojlim$ need not be exact. The situation gets more complicated given the result of Garavaglia [4] that equationally compact abelian groups can be characterised as being exactly those groups which, when taken as coefficients for Čech homology, guarantee the exactness of that theory.

Into this confused situation we throw another result. We show that given any inverse system of modules, M, there is an imbedding into a reduced power of that system, $D(M)$ and the derived functors of limit are zero on $D(M)$. The link with equational compactness is again complicated. For any abelian group G, its countable reduced power $G^{\mathbb{N}}/G^{(\mathbb{N})}$ is equationally compact but usually not linearly compact. For arbitrary rings R, there exist R-modules A such that $A^{\mathbb{N}}/A^{(\mathbb{N})}$ is not equationally compact as an R-module. It can even happen with A equationally compact. Gerstner [9] has shown that, for infinite I, the abelian group $\mathbb{Z}^I/\mathbb{Z}^{(I)}$ is equationally compact if and only if I is countable, but Werglorz [8] shows that an algebra is equationally compact if and only if it is a retract of each of its ultrapowers. Thus reduced powers are linked to equational compactness in a fundamental but rather confusing way (cf. Daley [2]).

The result we prove was first indicated by a result of Grossman [5]. His work connects up again with Čech and Steenrod homology [3]. (We have examined this topological connection in a separate paper [10].)

As a general reference for equational compactness we suggest Daley [2], for reduced powers Frayne, Morel and Scott [8] and for the derived functors of limit, Jensen [6]. I would like to thank Chris Jensen for his help in sorting out my confusion on the links between reduced powers, equational compactness and linear compactness.

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Notational remark. Throughout the note we shall assume a fixed ring R and 'module' will mean 'left R-module'. R-Mod will denote the category of modules.

1. Reduced and ultrapowers – definitions

Given a set I, a filter F on I and a module M, the reduced power of M over F is constructed as follows.

For each $i \in I$, let M_i be a copy of M. Define a relation θ_F on the product, $P = \prod \{M_i : i \in I\}$, by

$$\underline{a} \equiv \underline{b} \mod \theta_F \quad \text{if and only if} \quad \{i \in I : a(i) = b(i)\} \in F.$$

θ_F is an equivalence relation on P and there is a submodule N_F of P defined by

$$\underline{n} \in N_F \quad \text{if and only if} \quad \underline{0} \equiv \underline{n} \mod \theta_F$$

such that $P/\theta_F \cong P/N_F$. This module P/N_F is called the reduced power of M over F and will be denoted, M_F^I. The diagonal map gives an embedding $\Delta_F^I : M \to M_F^I$.

Example. If I is infinite and F is the cofinite filter on I (i.e. $A \in F$ if and only if $I \backslash A$ is finite) then $N_F = \sum_{i \in I} M_i$ and there is a short exact sequence

$$0 \to \sum M_i \to \prod M_i \to M_F^I \to 0 \quad (\text{or } 0 \to M^{(I)} \to M^I \to M_F^I \to 0).$$

If F is an ultrafilter (i.e. a maximal filter on I) then it is usual to say that M_F^I is the ultrapower of M over F.

Almost all the results that we will need on ultrapowers and reduced powers are mentioned in [2] or [8], further references can be found there. We mention again a result which gives the link with equational compactness:

M is equationally compact if and only if each Δ_F^I has a right inverse (i.e. there is a morphism $r_F^I : M_F^I \to M$ satisfying $r_F^I \Delta_F^I = \mathrm{id}_M$).

2. The complex $\Pi^* M$

Let $M : I \to R$-Mod be an inverse system of modules indexed by a directed set, I. (We consider I as a small category in the usual way.)

We adapt the construction given in Jensen [6, Ch. 4] to give not a complex but a cosimplicial module which can be used to calculate $\varprojlim^{(q)} M$ (c.f. Bousfield–Kan [1, Ch. XI]). $N(I)$ will denote the nerve of I, that is the simplicial set with, as typical n-simplex, an $(n+1)$-tuple of elements of I,

$$u = i_0 \xleftarrow{\alpha_1} i_1 \xleftarrow{\alpha_2} \cdots \xleftarrow{\alpha_n} i_n$$

4. Applications

We offer here some slight algebraic applications of our result.

1. If M is any inverse system and $D(M)$ is as in the theorem of Section 3 there is a diagonal morphism

$$\Delta : M \to D(M)$$

of inverse systems, if this morphism splits then clearly $\varprojlim^{(q)} M = 0$ for all $q > 0$.

2. For any $M, D(M)$ as in Section 3, we can construct a resolution of M

$$0 \to M \xrightarrow{\Delta} D^{(*)}(M)$$

in the obvious way, $D^{(0)}(M) = D(M)$, $D^{(1)}(M) = D(\mathrm{Coker}\,\Delta)$, etc. Applying $\varprojlim$ one obtains a complex, $\varprojlim D^{(*)}(M)$ such that

$$H^q(\varprojlim D^{(*)}(M)) = \varprojlim^{(q)} M.$$

It does not seem to have been noticed before that such a complex can be used to calculate $\varprojlim^{(q)} M$.

3. [For $R = \mathbb{Z}$]. From observation 2, together with the fact that ultrapowers are equationally compact (and hence pure injective) it follows that if F is an ultrafilter on I_0, for each i, $D^{(*)}(M(i))$ is a pure injective resolution of $M(i)$. This leads to the following spectral sequences. Taking $\Pi^* D^{(*)}(M)$ gives a bicomplex. Let A be any abelian group and consider the bicomplex

$$B^{p,q} = \mathrm{Hom}(A, \Pi^p D^{(q)}(M))$$

The two spectral sequences of B^{**} are

$${}_{\mathrm{I}}E_2^{p,q} = \begin{cases} H^q(\mathrm{Hom}(A, \varprojlim D^{(*)}(M)), & p = 0, \\ 0, & p \neq 0, \end{cases}$$

$${}_{\mathrm{II}}E_2^{p,q} = \begin{cases} \varprojlim^{(q)} \mathrm{Pext}_R^p(A, M(i)), & p, q \geq 0,\ p = 0, 1, \\ 0, & \text{otherwise.} \end{cases}$$

(The second spectral sequence occurs since

$$B^{p,q} \cong \Pi^p \,\mathrm{Hom}(A, D^{(q)}(M));$$

$\mathrm{Pext}_R^{(p)}$ denotes the group of pure extensions of length p (cf. Jensen [6]).)

If A is pure projective, for instance, of finite type, we have $\mathrm{Pext}_R^p(A, M(i)) = 0$ for $p > 0$ and the ${}_{\mathrm{II}}E$ sequence collapses to give

$$H^q(\mathrm{Hom}(A, \varprojlim D^{(*)}(M))) \cong \varprojlim^{(q)} \mathrm{Hom}(A, M(i)).$$

The same collapse occurs if each $M(i)$ is pure-injective (i.e. equationally compact).

One can, of course, obtain more information on $H^q(\mathrm{Hom}(A, \varprojlim D^*(M)))$ using the fact that $H^q(\varprojlim D^*(M)) \cong \varprojlim^{(q)} M$ and one of the hyperhomology spectral sequences.

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